

Singularities: Analytic Combinatorics for Counting

Yuxuan Li

April 5, 2026

Abstract

This report introduces analytic combinatorics through singularity analysis. The main idea is that the asymptotic behaviour of a sequence is determined by the singularities of its generating function. We explain how the radius of convergence and dominant singularities control exponential growth, and how the local type of a singularity yields more precise asymptotic information. We then discuss meromorphic asymptotics and the transfer theorem, illustrated by examples such as surjections, alternating permutations, Catalan numbers, and rooted labeled trees.

1 Analytic Combinatorics

Analytic combinatorics studies generating functions as complex analytic objects. The locations and types of their singularities translate into asymptotic information about coefficients. In this section we extend the point of view to more general generating functions.

1.1 Radius of convergence and dominant singularities

Definition 1.1 (Limit superior). *For a sequence (x_n) , the quantity $\limsup_{n \rightarrow \infty} x_n$ is the supremum of all subsequential limits. In particular, it always exists in $[-\infty, \infty]$.*

Theorem 1.2 (Root Test / Cauchy–Hadamard Theorem [3]). *Let*

$$\rho = \limsup_{n \rightarrow \infty} |a_n|^{1/n}, \quad R = \begin{cases} 1/\rho & (0 < \rho < \infty), \\ \infty & (\rho = 0), \\ 0 & (\rho = \infty). \end{cases}$$

Then the power series $F(z) = \sum_{n \geq 0} a_n z^n$ converges for $|z| < R$ and diverges for $|z| > R$.

Definition 1.3 (Radius of convergence). *The number R appearing in the Cauchy–Hadamard theorem is called the **radius of convergence** of the power series $F(z)$.*

Example 1.4. *The radius of convergence can often be computed directly from the coefficients.*

- *Since $\lim_{n \rightarrow \infty} (2^n)^{1/n} = 2$, the series*

$$\sum_{n \geq 0} 2^n z^n = \frac{1}{1 - 2z}$$

has radius of convergence $R = 1/2$.

- *Since*

$$\lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} = \infty,$$

the series $\sum_{n \geq 0} n! z^n$ has radius of convergence $R = 0$.

The radius of convergence already encodes coarse asymptotic information. In the examples above, it was obtained directly from explicit coefficient formulas. The natural question is what one can do when no such formula is known.

A central insight of analytic combinatorics is that the radius of convergence, and more generally the growth of the coefficients, can often be read off from the analytic structure of the generating function itself. The rational case already illustrates this.

Example 1.5. *Suppose*

$$F(z) = \frac{G(z)}{H(z)}$$

is rational, where G and H are coprime polynomials. Then the exponential growth of $[z^n]F(z)$ is determined by the roots of the denominator H : more precisely, it is governed by the reciprocal of the smallest modulus of a root of H .

To move beyond rational functions, we need the notion of a *singularity*.

Definition 1.6 (Analytic). *A function $F(z)$ is **analytic** at $z = a$ if it can be expanded in a convergent power series around a . Equivalently, it has no singularity there.*

Definition 1.7 (Singularity). *A **singularity** of a complex function is a point where the function fails to be analytic.*

For instance, a function may fail to be analytic because its defining expression involves division by zero, or because a square root or logarithm becomes singular. A fully rigorous treatment requires the language of analytic continuation, which we will not develop here.

Theorem 1.8 (Cauchy [3]). *If*

$$F(z) = \sum_{n \geq 0} f_n z^n$$

has finite positive radius of convergence R , then R is the minimum modulus of the singularities of F in the complex plane.

Because we are mainly interested in generating functions with nonnegative coefficients, we can say more.

Theorem 1.9 (Vivanti–Pringsheim Theorem [3]). *If*

$$F(z) = \sum_{n \geq 0} f_n z^n$$

and $f_n \geq 0$ for all n , then the point $z = R$, where R is the radius of convergence, is itself a singularity of $F(z)$. In particular, R is a singularity of minimal modulus.

Coarse growth from the radius of convergence. If

$$F(z) = \sum_{n \geq 0} f_n z^n$$

has radius of convergence $R > 0$, then the Cauchy–Hadamard formula implies

$$\limsup_{n \rightarrow \infty} |f_n|^{1/n} = \frac{1}{R}.$$

Thus the coefficients grow, at the coarsest level, like R^{-n} . In other words, locating the nearest singularity gives a first approximation to the size of f_n , even before one knows any subexponential factors. When the coefficients are nonnegative, the Vivanti–Pringsheim theorem identifies the dominant singularity on the positive real axis, which is especially useful in combinatorial applications.

Definition 1.10 (Dominant singularity). A *dominant singularity* of $F(z)$ is a singularity of minimal modulus.

Example 1.11. The location of the first positive real singularity correctly predicts the exponential growth in a number of familiar cases.

- If

$$F(z) = \frac{1}{1 - 2z},$$

then the dominant singularity is at $z = 1/2$, so the coefficients grow like 2^n .

- If

$$F(z) = \frac{1 - \sqrt{1 - 4z}}{2z},$$

then the dominant singularity is at $z = 1/4$, so the coefficients have exponential scale 4^n . In fact,

$$[z^n]F(z) \sim \frac{4^n}{\sqrt{\pi} n^{3/2}}.$$

- The exponential generating function for surjections is

$$F(z) = \frac{1}{2 - e^z}.$$

The first positive real singularity is at $z = \log 2$, so

$$[z^n]F(z)$$

has exponential scale $(\log 2)^{-n}$.

1.2 Meromorphic asymptotics

To pass from coarse exponential growth to sharper asymptotics, one must study not only the location of the dominant singularity, but also its *type*. The following theorem gives precise asymptotics in the simple-pole case.

Theorem 1.12 (Meromorphic Asymptotics [3]). Let $F(z) = G(z)/H(z)$, where G and H are analytic in $|z| \leq B$. Suppose $H(z)$ has exactly one simple zero w with $|w| < B$, and suppose $G(w) \neq 0$. Then, as $n \rightarrow \infty$,

$$[z^n]F(z) = -\frac{G(w)}{wH'(w)}w^{-n} + O(B^{-n}).$$

Example 1.13. Find asymptotics for the number s_n of surjections on an n -element set.

Solution. The exponential generating function is

$$F(z) = \frac{1}{2 - e^z}.$$

Here $G(z) = 1$ and $H(z) = 2 - e^z$. The dominant singularity is the simple zero $w = \log 2$. Moreover,

$$G(w) = 1, \quad H'(w) = -e^w = -2.$$

The theorem gives

$$[z^n]F(z) \sim -\frac{1}{w(-2)}w^{-n} = \frac{1}{2\log 2}(\log 2)^{-n}.$$

Since F is an exponential generating function,

$$s_n = n![z^n]F(z) \sim \frac{n!}{2(\log 2)^{n+1}}.$$

On the other hand, we can find that

$$F(z) = \frac{1}{2} \cdot \frac{1}{1 - e^z/2} = \sum_{m \geq 0} \left(\frac{1}{2} \sum_{k \geq 0} \frac{k^m}{2^k} \right) \frac{z^m}{m!}$$

This agrees with the asymptotic formula obtained above, confirming that the dominant singularity at $z = \log 2$ determines the leading growth of the coefficients. \square

Example 1.14. The exponential generating function for alignments ($\mathcal{A} = \text{SEQ}(\text{CYC}(z))$) is

$$A(z) = \frac{1}{1 + \log(1 - z)}.$$

The equation

$$1 + \log(1 - z) = 0$$

has solution $z = 1 - e^{-1}$, which is the dominant singularity. A meromorphic analysis gives

$$[z^n]A(z) \sim \frac{1}{e}(1 - e^{-1})^{-n-1}.$$

Hence the counting sequence satisfies

$$a_n = n![z^n]A(z) \sim \frac{n!}{e(1 - e^{-1})^{n+1}}.$$

Example 1.15. Exercise. Show that (André's problem [1]) the exponential generating function for alternating permutations of odd size is

$$T(z) = \tan z.$$

Use meromorphic asymptotics to derive the leading growth of the Euler numbers.

Proof. Let t_n denote the number of alternating permutations of $[2n + 1]$ of the form

$$\pi_1 < \pi_2 > \pi_3 < \pi_4 > \cdots < \pi_{2n} > \pi_{2n+1},$$

and define

$$T(z) = \sum_{n \geq 0} t_n \frac{z^{2n+1}}{(2n+1)!}.$$

We first derive a recurrence for t_n . In such a permutation, the maximum entry $2n + 1$ must occur at an even position, say position $2k$ with $1 \leq k \leq n$. Removing this maximum splits the permutation into a left part of length $2k - 1$ and a right part of length $2n - 2k + 1$, each of which is again alternating after standardization. Hence

$$t_n = \sum_{k=1}^n \binom{2n}{2k-1} t_{k-1} t_{n-k} \quad (n \geq 1),$$

with $t_0 = 1$. Now differentiate $T(z)$:

$$T'(z) = \sum_{n \geq 0} t_n \frac{z^{2n}}{(2n)!}.$$

On the other hand,

$$T(z)^2 = \left(\sum_{i \geq 0} t_i \frac{z^{2i+1}}{(2i+1)!} \right) \left(\sum_{j \geq 0} t_j \frac{z^{2j+1}}{(2j+1)!} \right),$$

so the coefficient of $z^{2n}/(2n)!$ in $T(z)^2$ is also

$$\sum_{k=1}^n \binom{2n}{2k-1} t_{k-1} t_{n-k} = t_n \quad (n \geq 1).$$

Therefore

$$T'(z) = 1 + T(z)^2,$$

since the constant term of $T'(z)$ is $t_0 = 1$, while the constant term of $T(z)^2$ is 0. Finally, $T(0) = 0$, and $\tan z$ satisfies

$$\frac{d}{dz} \tan z = 1 + \tan^2 z, \quad \tan 0 = 0.$$

By uniqueness of solutions to this differential equation,

$$T(z) = \tan z.$$

To derive the leading growth, note that

$$\tan z = \frac{\sin z}{\cos z},$$

so the poles occur where $\cos z = 0$, namely at

$$z = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}.$$

The dominant poles are therefore

$$w_1 = \frac{\pi}{2}, \quad w_2 = -\frac{\pi}{2}.$$

Thus the theorem does not apply directly, since the dominant pole is not unique. We apply it to each simple pole separately and add the two contributions. For $w_1 = \pi/2$,

$$G(w_1) = 1, \quad H'(w_1) = -1,$$

so

$$-\frac{G(w_1)}{w_1 H'(w_1)} w_1^{-n} = \frac{2}{\pi} \left(\frac{2}{\pi}\right)^n.$$

For $w_2 = -\pi/2$,

$$G(w_2) = -1, \quad H'(w_2) = 1,$$

so

$$-\frac{G(w_2)}{w_2 H'(w_2)} w_2^{-n} = -\frac{2}{\pi} \left(-\frac{2}{\pi}\right)^n.$$

Hence

$$[z^n] \tan z \sim \frac{2}{\pi} \left(\frac{2}{\pi}\right)^n - \frac{2}{\pi} \left(-\frac{2}{\pi}\right)^n.$$

In particular, for odd n

$$[z^n] \tan z \sim 2 \left(\frac{2}{\pi}\right)^{n+1}.$$

Since

$$\tan z = \sum_{m \geq 0} t_m \frac{z^{2m+1}}{(2m+1)!},$$

it follows, by taking $n = 2m + 1$, that

$$\frac{t_m}{(2m+1)!} \sim 2 \left(\frac{2}{\pi}\right)^{2m+2}.$$

Hence

$$t_m \sim 2 \left(\frac{2}{\pi}\right)^{2m+2} (2m+1)!.$$

□

The meromorphic theorem above treats the case in which the dominant singularity is a simple pole. In that setting, once the dominant pole w is identified, the coefficient asymptotics follow directly and take the form

$$[z^n]F(z) \sim C w^{-n}.$$

Thus the location of the singularity determines the exponential growth, while the fact that the singularity is a pole determines that there is no additional polynomial correction term.

However, many generating functions arising in combinatorics are not meromorphic at their dominant singularities. Typical examples involve square roots, logarithms, or more general fractional powers. In such cases, the dominant singularity still controls the exponential term, but the *local shape* of the function near that singularity contributes an additional subexponential factor, usually a power of n .

The transfer theorem is the basic tool that makes this principle precise. Roughly speaking, it says that if one knows the local expansion of $F(z)$ near its dominant singularity ρ , then one can “transfer” that local singular behaviour directly into an asymptotic formula for the coefficients $[z^n]F(z)$. In this sense, the meromorphic theorem may be viewed as the simplest special case of a more general singularity analysis framework.

Theorem 1.16 (Transfer Theorem / Singularity Analysis [2, 3]). *Let*

$$F(z) = \sum_{n \geq 0} f_n z^n$$

and suppose that $\rho > 0$ is the unique dominant singularity of F , and that F is analytic in a dented neighborhood of ρ . Assume that, as $z \rightarrow \rho$,

$$F(z) = a_0 + a_1 \left(1 - \frac{z}{\rho}\right)^\beta + O\left(\left(1 - \frac{z}{\rho}\right)^{\beta+1}\right),$$

where $\beta \notin \{0, 1, 2, \dots\}$. Then

$$[z^n]F(z) \sim \frac{a_1}{\Gamma(-\beta)} n^{-\beta-1} \rho^{-n}.$$

Equivalently, if

$$F(z) = a_0 + a_1 \left(1 - \frac{z}{\rho}\right)^{-\alpha} + O\left(\left(1 - \frac{z}{\rho}\right)^{-\alpha+1}\right),$$

with $-\alpha \notin \{1, 2, 3, \dots\}$, then

$$[z^n]F(z) \sim \frac{a_1}{\Gamma(\alpha)} n^{\alpha-1} \rho^{-n}.$$

The significance of the Transfer Theorem is that it converts local analytic information at the dominant singularity into coefficient asymptotics. In other words, once the behavior of $F(z)$ near its dominant singularity ρ is known, the asymptotic form of $[z^n]F(z)$ can be read off directly. The following examples demonstrate how this principle is applied in practice.

Example 1.17 (Catalan numbers). *Let*

$$C(z) = \frac{1 - \sqrt{1 - 4z}}{2z}.$$

The dominant singularity is

$$\rho = \frac{1}{4},$$

since the singularity occurs when $1 - 4z = 0$. Near $z = \rho$, we rewrite

$$C(z) = \frac{1 - \sqrt{1 - 4z}}{2z}$$

in the form

$$C(z) = 2 - 2(1 - 4z)^{1/2} + O(1 - 4z).$$

Thus the singular term is

$$-2 \left(1 - \frac{z}{\rho}\right)^{1/2}, \quad \rho = \frac{1}{4}.$$

So in the transfer theorem we take

$$a_1 = -2, \quad \beta = \frac{1}{2}, \quad \rho = \frac{1}{4}.$$

Hence

$$[z^n]C(z) \sim \frac{a_1}{\Gamma(-\beta)} n^{-\beta-1} \rho^{-n} = \frac{-2}{\Gamma(-1/2)} n^{-3/2} 4^n.$$

Since

$$\Gamma(-1/2) = -2\sqrt{\pi},$$

we obtain

$$[z^n]C(z) \sim \frac{4^n}{\sqrt{\pi} n^{3/2}}.$$

Example 1.18 (Rooted labeled trees). *Let t_n be the number of rooted labeled trees on n vertices, and let*

$$T(z) = \sum_{n \geq 1} t_n \frac{z^n}{n!}.$$

A standard combinatorial decomposition gives the functional equation

$$T(z) = ze^{T(z)}.$$

We determine the dominant singularity by rewriting this as

$$z = Te^{-T}.$$

The inverse ceases to be analytic when

$$\frac{d}{dT}(Te^{-T}) = e^{-T}(1 - T) = 0,$$

so the critical point is at $T = 1$. Hence the dominant singularity is

$$\rho = 1 \cdot e^{-1} = \frac{1}{e}.$$

To obtain the local expansion, write $T = 1 - u$. Then

$$z = (1 - u)e^{-(1-u)} = \frac{1}{e}(1 - u)e^u = \frac{1}{e} \left(1 - \frac{u^2}{2} + O(u^3) \right).$$

Therefore

$$1 - \frac{z}{\rho} = \frac{u^2}{2} + O(u^3),$$

so

$$u \sim \sqrt{2 \left(1 - \frac{z}{\rho} \right)}.$$

Since $T = 1 - u$, we obtain

$$T(z) = 1 - \sqrt{2} \left(1 - \frac{z}{\rho} \right)^{1/2} + O \left(1 - \frac{z}{\rho} \right).$$

Thus, in the transfer theorem, we take

$$a_1 = -\sqrt{2}, \quad \beta = \frac{1}{2}, \quad \rho = \frac{1}{e}.$$

It follows that

$$[z^n]T(z) \sim \frac{a_1}{\Gamma(-1/2)} n^{-3/2} \rho^{-n} = \frac{e^n}{\sqrt{2\pi} n^{3/2}}.$$

Equivalently,

$$\frac{t_n}{n!} \sim \frac{e^n}{\sqrt{2\pi} n^{3/2}},$$

and hence

$$t_n \sim \frac{n! e^n}{\sqrt{2\pi} n^{3/2}}.$$

Using Stirling's formula, this simplifies to

$$t_n \sim n^{n-1}.$$

1.3 The principles of analytic combinatorics

The examples above illustrate the two main guiding principles of analytic combinatorics.

First Principle. The location of the dominant singularity determines the exponential growth of the coefficients. If the dominant singularity is at modulus R , then the coefficients have exponential scale R^{-n} .

Second Principle. The *type* of the dominant singularity determines the subexponential factor. Poles, algebraic singularities, and logarithmic singularities lead to different polynomial or logarithmic corrections.

In practice, one first identifies the dominant singularity of the generating function and then studies its local form near that singularity. Standard transfer theorems convert this local analytic information into coefficient asymptotics. This is the basic mechanism that underlies the later connection between asymptotic enumeration and Boltzmann sampling.

References

- [1] Désiré André. Sur les permutations alternées. *Journal de Mathématiques Pures et Appliquées*, 7:167–184, 1881. Archived copy via the Internet Archive.
- [2] Philippe Flajolet and Andrew M. Odlyzko. Singularity analysis of generating functions. *SIAM Journal on Discrete Mathematics*, 3(2):216–240, 1990.
- [3] Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics*. Cambridge University Press, Cambridge, 2009.
- [4] Stephen Melczer. An invitation to enumeration, 2024.